The Directed Network Design Problem with Relays

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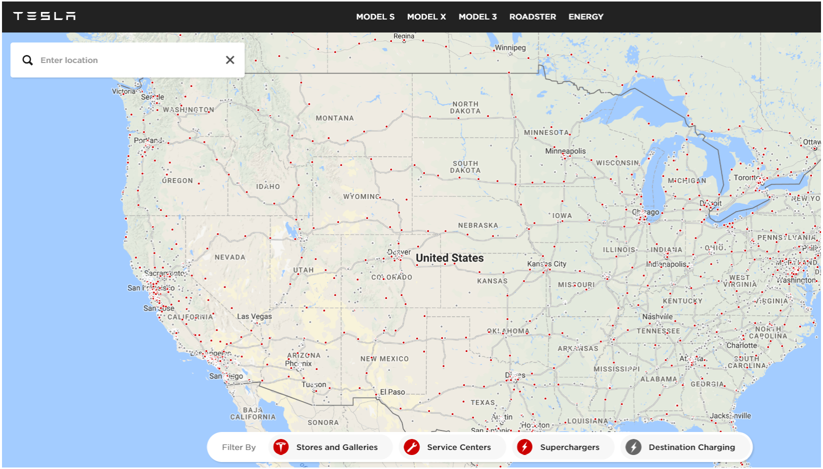
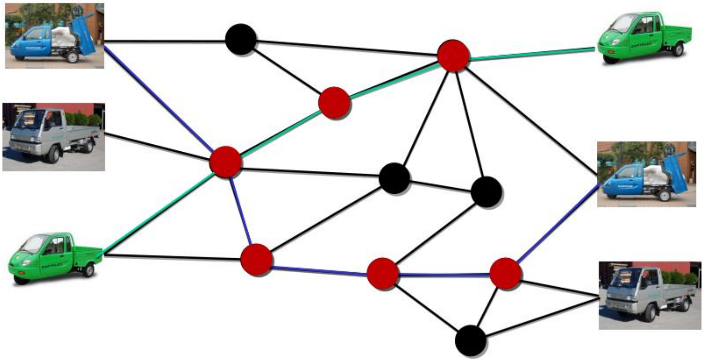
Network Design with Relays

- Models network design problems in transportation and telecommunication.
- Freight transportation networks: for long haul distance trips, relay points are set along the paths for the exchange of drivers, trucks and trailers.
- Telecommunication networks: optical signal deteriorates after traversing a certain distance, and has to be re-amplified, i.e., regenerator devices need to be installed.
- E-mobility networks: batteries of EVs need to be recharged after a certain distance, hence charging stations need to be placed in the network.

Tesla Supercharger Network (≈1200 stations)



Network Design with Relays



1. Network Design: Build the network or augment the existing one.
2. Location: Where to place relays, and how many?
3. Routing: How to route each commodity from its source to destination?

PROBLEM DEFINITION

Directed Network Design with Relays

Given:

- directed graph $G = (V, A)$
- relay placement costs $c \colon V \to \mathbb{Z}_{>0}$
- arc costs $w \colon A \to \mathbb{Z}_{\geq 0}$ and arc lengths $d \colon A \to \mathbb{Z}_{\geq 0}$
- set $\mathcal{K}$ of O-D pairs (commodities)
- distance limit $\lambda_{\max} \in \mathbb{Z}_{>0}$

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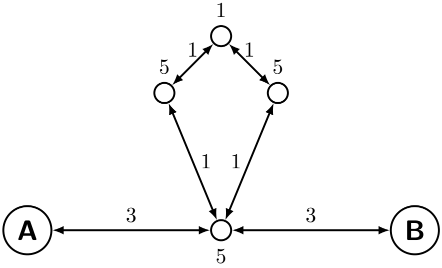
- distance limit $\lambda_{\max} \in \mathbb{Z}_{>0}$

Goal:

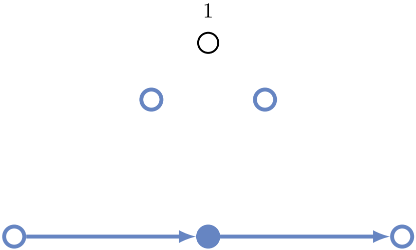
- install a subset of relays and arcs of minimum cost s.t. there exists a feasible simple path for each O-D pair from $\mathcal{K}$.
- an O-D path P is feasible if each subpath of P which is longer than $\lambda_{\max}$ contains a relay

Example — Symmetric Instance

- $\lambda_{\max} = 5$, $\mathcal{K} = \{(A, B)\}$



Instance



Acyclic Solution (cost=5)

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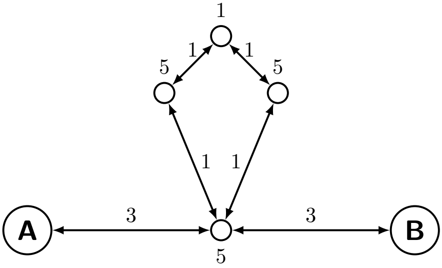

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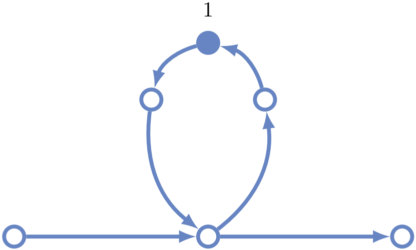

Cyclic Solution (cost=1)

Previous Work

Undirected NDPR:

- Cabral et al. (2007): Set-covering formulation (each column is an O-D path, including relays)
- Heuristics: VNS, Xiao and Konak (2017), tabu search, Lin et al. (2014), GAs, Kulturel-Konak and Konak (2008); Konak (2012)
- Exact algorithms based on B&P&C (columns are segments between the relays):
    - Yıldız et al. (2018)
    - Leitner et al. (2018)

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Directed NDPR:

- Introduced in Li et al. (2012), exact, 2 models:
  - compact Node-Arc model
  - Set-Covering model (similar to Cabral et al. (2007)) $\Rightarrow$ B&P
- Heuristic: Li et al. (2017)

Our contribution:

Directed NDPR:

- New models based on layered graphs (distance-expanded graphs):
  - multi-commodity flows
  - cut-sets
- Branch-and-Cut (B&C) algorithms for both models
- Both B&C significantly outperform the previous state-of-the-art from Li et al. (2012)

A BASIC FORMULATION

Node Arc Formulation from Li et al. (2012)

$$b_i^k = \begin{cases} 1 & \text{if } k = (i, v) \\ -1 & \text{if } k = (u, i) \\ 0 & \text{otherwise} \end{cases} \quad (u, v) \in \mathcal{K}$$

$v_i^k =$ distance of node i from the preceeding relay for commodity k.

$$y_i = \begin{cases} 1 & \text{if relay is installed at node } i \\ 0 & \text{otherwise} \end{cases} \quad i \in V$$

$$x_a = \begin{cases} 1 & \text{if arc } a \text{ is installed} \\ 0 & \text{otherwise} \end{cases} \quad a \in A$$

Node Arc Formulation from Li et al. (2012)

$$\text{(NA)} \quad \min \sum_{i \in V} c_i y_i + \sum_{a \in A} w_a x_a$$

$$\sum_{a \in \delta+(i)} f_a^k - \sum_{a \in \delta^-(i)} f_a^k = b_i^k \qquad \forall k \in \mathcal{K}, \forall i \in V \qquad (1)$$

$$v_i^k + d_{(i,j)} f_{(i,j)}^k - \lambda_{\max}(1 - f_{(i,j)}^k + y_j) \leq v_j^k \qquad \forall k \in \mathcal{K}, \forall (i,j) \in A \qquad (2)$$

$$v_i^k + d_{(i,j)} f_{(i,j)}^k \leq \lambda_{\max} \qquad \forall k \in \mathcal{K}, \forall (i,j) \in A \qquad (3)$$

$$f_a^k \leq x_a \qquad \forall k \in \mathcal{K}, \forall a \in A \qquad (4)$$

$$0 \leq v_i^k \leq \lambda_{\max}(1 - y_i) \qquad \forall k \in \mathcal{K}, \forall i \in V \qquad (5)$$

$$v_u^{u,v} = 0 \qquad \forall (u,v) \in \mathcal{K} \qquad (6)$$

$$f_a^k \in \{0,1\} \qquad \forall k \in \mathcal{K}, \forall a \in A \qquad (7)$$

$$y_i \in \{0,1\} \qquad \forall i \in V \qquad (8)$$

$$0 \leq x_a \leq 1 \qquad \forall a \in A \qquad (9)$$

MODELS ON LAYERED GRAPHS

Solution Structure

Set S of commodity sources, set T^u of targets of source u

Single-source case:

If $S = \{u\}$, there exists an optimal solution which is a Steiner arborescence rooted at u, with leaves from T^u. Each O-D path in this tree must be made feasible by installing some relays (when needed).

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Multiple sources:

An optimal solution is a union of Steiner arborescences rooted at u, with required placement of relays when needed.

Steiner arborescence: rooted subtree connecting a given set of terminals.

Example

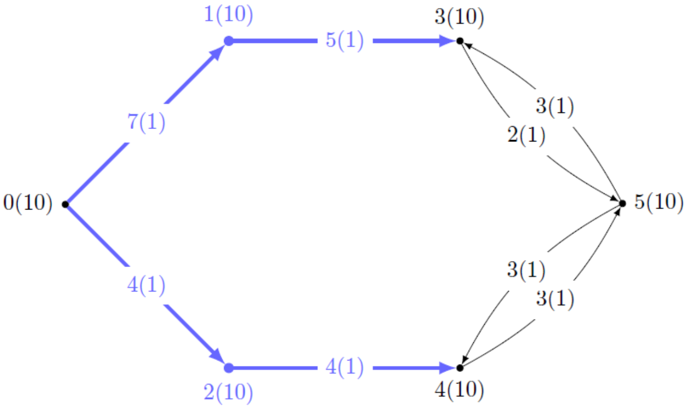
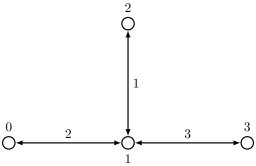


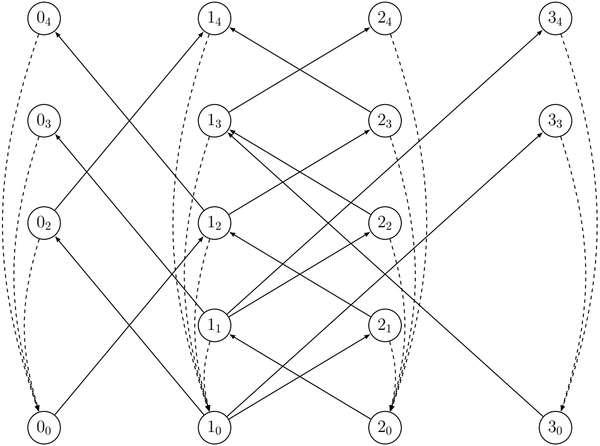
Figure 1: Example instance with two commodities $\mathcal{K} = \{(0,3),(0,4)\}$ and $\lambda_{\max} = 7$. Arc distances are provided next to the arcs, relay and arc costs are given in parentheses. Relays and arcs used in the optimal solution are marked bold and blue.

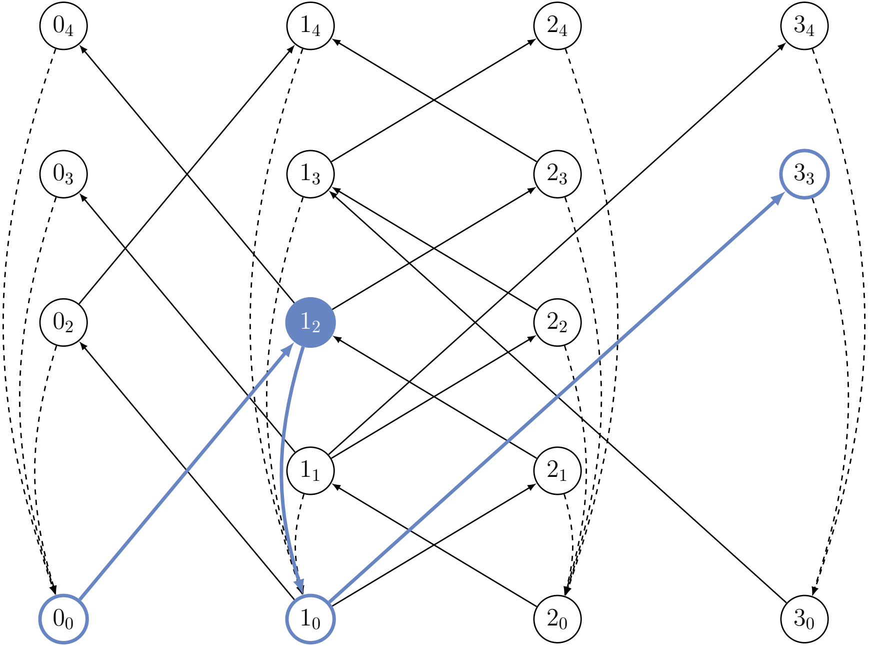
How to integrate the fact that on some nodes of the Steiner tree relays have to be installed?

Basic Idea

- Create node copies according to feasible distances at which a node can be reached
- Embed Steiner trees into this network, for each source u

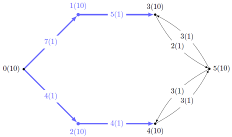






0_4
1_4
2_4
3_4
0_3
1_3
2_3
3_3
0_2
1_2
2_2
1_1
2_1
0_0
1_0
2_0
3_0

Solution (Single Source)



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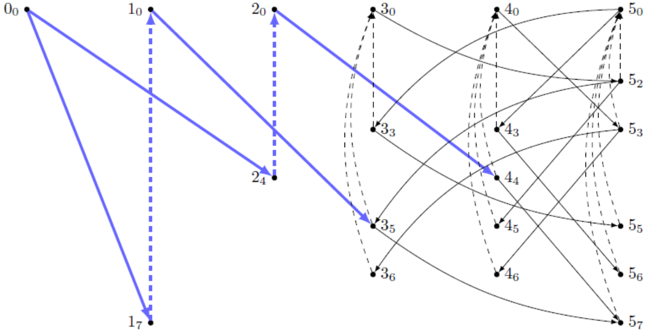

Solution: Steiner tree rooted at 0, each target reached at some layer.

Layered Graph Models

Model Name	Connectivity	Aggregation	Type
L-CUT	cutsets	per source	B&C
L-MCF	multi-commodity flow	none	pseudo-compact B&C

- L-CUT:

$$z_a^u \in \{0,1\} \qquad \forall u \in S, \forall a \in A_{\mathrm{L}}^u$$

- L-MCF:

$$f_a^{uv} \in \{0,1\} \qquad \forall (u,v) \in \mathcal{K}, \forall a \in A_{\mathrm{L}}^u$$

Layered Cut Model

$$\text{(L-CUT)} \quad \min \sum_{i \in V} c_i y_i + \sum_{a \in A} w_a x_a$$

Ensure connectivity between the source u and a copy of $v \in T^u$:

$$\sum_{a \in \delta^-(W)} z_a^u \geq 1 \qquad \forall u \in S, \forall v \in T^u, \; \{v_l | v_l \in V_{\mathrm{L}}^u\} \subseteq W \subset V_{\mathrm{L}}^u, \; u \notin W$$

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Indegree of a node v over all layers is at most one for $i \notin T^u$, and exactly one for $i \in T^u$.

$$\sum_{i_l \in V_{\mathrm{L}}^u} \sum_{a \in \delta^-(i_l), a \notin A_{\mathrm{L}}^r} z_a^u \leq 1 \qquad \forall u \in S, \forall i \notin T^u, i \neq u$$

$$\sum_{i_l \in V_{\mathrm{L}}^u} \sum_{a \in \delta^-(i_l), a \notin A_{\mathrm{L}}^r} z_a^u = 1 \qquad \forall u \in S, \forall i \in T^u$$

Layered Cut Model (cont.)

Vertical arcs linked to relays:

$$\sum_{(i_l,i_0)\in A^u_{\mathrm{L}}} z^u_{(i_l,i_0)} \leq y_i \qquad \forall u \in S, \forall i \in V$$

Each $(i,j) \in A$ can be used in at most one layer

$$\sum_{(i_l,j_m)\in A^u_{\mathrm{L}}} z^u_{(i_l,j_m)} \leq x_{(i,j)} \qquad \forall u \in S, \forall (i,j) \in A$$

Layered MCF Model: No linking with z_a^u needed

$$
\begin{aligned}
\min \; & \sum_{i \in V} c_i y_i + \sum_{a \in A} w_a x_a && && \text{(2a)} \\
\text{s.t.} \; & \sum_{a \in \delta^+(u_0)} f_a^{uv} = 1 && \forall (u,v) \in \mathcal{K} && \text{(2b)} \\
& \sum_{a \in \delta^-(i_l)} f_a^{uv} - \sum_{a \in \delta^+(i_l)} f_a^{uv} = 0 && \forall (u,v) \in \mathcal{K}, \forall i_l \in V_{\mathrm{L}} : i \notin \{u,v\} && \text{(2c)} \\
& \sum_{v_l \in V_{\mathrm{L}}} \sum_{a \in \delta^-(v_l) \setminus A_{\mathrm{L}}^{\mathrm{r}}} f_a^{uv} = 1 && \forall (u,v) \in \mathcal{K} && \text{(2d)} \\
& \sum_{i_l \in V_{\mathrm{L}}} \sum_{a \in \delta^-(i_l) \setminus A_{\mathrm{L}}^{\mathrm{r}}} f_a^{uv} \le 1 && \forall (u,v) \in \mathcal{K}, \forall i \in V \setminus \{u,v\} && \text{(2e)} \\
& \sum_{a=(i_l,i_0) \in A_{\mathrm{L}}^{\mathrm{r}}} f_a^{uv} \le y_i && \forall (u,v) \in \mathcal{K}, \forall i \in V && \text{(2f)} \\
& \sum_{a=(i_l,j_m) \in A_{\mathrm{L}}^{\mathrm{a}}} f_a^{uv} \le x_{ij} && \forall (u,v) \in \mathcal{K}, \forall (i,j) \in A && \text{(2g)} \\
& y_i \in \{0,1\} && \forall i \in V && \text{(2h)} \\
& x_a \in \{0,1\} && \forall a \in A && \text{(2i)} \\
& 0 \le f_a^u \le 1 && \forall (u,v) \in \mathcal{K}, \forall a \in A_{\mathrm{L}} && \text{(2j)}
\end{aligned}
$$

Comparing the strength of the two models

Theorem

Formulations L-MCF and L-CUT are equally strong, i.e., the LP-relaxation values of the two models coincide.

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Formulations L-MCF and L-CUT are equally strong, i.e., the LP-relaxation values of the two models coincide.

- Further strengthening is possible for L-CUT
- There are symmetries induced by the layered graph

L-CUT: Strengthening Cuts

Flow-balance

In-degree $\leq$ out-degree for every non-target node in LG:

$$\sum_{a \in \delta^-(i_l)} z_a^u \leq \sum_{a \in \delta^+(i_l)} z_a^u \qquad \forall u \in S, \forall i_l \in V_{\mathrm{L}}^u, i \notin T^u \cup \{u\}$$

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Symmetry Breaking

The same optimal solution may have multiple embeddings in the LG (2 commodities share a subpath, and only one of them uses a relay). Force that in routing path, if relay is installed, it must be used:

$$\sum_{(i_l,j_m) \in A_{\mathrm{L}}^u : l>0 \wedge m>0} z_{(i_l,j_m)}^u \leq M_i^u \cdot (1 - y_i) \qquad \forall u \in S, \forall i \in V, \; i \neq u$$

$$M_i^u = \begin{cases} \min(|T^u|, |\delta^+(i)|) & i \notin T^u \\ \min(|T^u| - 1, |\delta^+(i)|) & i \in T^u \end{cases}$$

L-MCF

Symmetry Breaking

$$\sum_{(i_l, j_m) \in A^u_{\mathrm{L}} : l > 0 \wedge m > 0} f^{uv}_{(i_l, j_m)} \leq 1 - y_i \qquad \forall (u,v) \in \mathcal{K}, \forall i \in V, \; i \neq u$$

COMPUTATIONAL RESULTS

Implementation Details

- preprocessing
  - remove "2-cycle arcs" for $|\delta^+(i)| \leq 1$ or $|\delta^-(i)| \leq 1$ vertices
  - remove unreachable vertices including their outgoing arcs
- initial heuristic based on Cabral et al.'s CH1
- original graph cuts to improve convergence speed of the cut model

$$\sum_{a \in \delta^-(W)} x_a \geq 1 \qquad \forall (u,v) \in \mathcal{K}, W \subset V, \; u \notin W, v \in W \tag{10}$$

- nested back cuts
- cost-based branching priorities

Four Settings

- NA: node-arc based model by Li et al. (2012)
- L-MCF
- L-CUT-d, dynamic (separation below)
- L-CUT-s, static (steps 2. and 3. skipped)

Separation

1. separate cut-set inequalities on the original graph
2. separate flow-balance constraints
3. separate two-cycle inequalities
4. if no flow-balance constraints and two-cycle inequalities added, separate cut-sets on the LG

Quality of Lower Bounds

Each line is average over 10 instances (from Cabral et al. (2007), EJOR)

Instance	Properties				LP gap [%]		
	$\|V\|$	$\|E\|$	λ_{max}	$\|\mathcal{K}\|$	L_{MCF}	L_{CUT}	NA
04A05B70L05K	20	62	70	5	0.2	**0.0**	27.6
04A05B70L10K	20	62	70	10	0.2	**0.0**	35.0
05A05B70L05K	25	80	70	5	0.8	**0.0**	31.4
05A05B70L10K	25	80	70	10	0.1	**0.0**	34.4
06A05B70L05K	30	98	70	5	0.5	**0.0**	36.8
06A05B70L10K	30	98	70	10	0.6	**0.0**	34.9
07A05B70L05K	35	116	70	5	0.1	**0.0**	40.5
07A05B70L10K	35	116	70	10	0.7	**0.1**	40.6
08A05B70L05K	40	134	70	5	0.1	**0.0**	45.1
08A05B70L10K	40	134	70	10	1.0	**0.1**	40.2
09A05B70L05K	45	152	70	5	0.1	**0.0**	42.9
09A05B70L10K	45	152	70	10	0.7	**0.0**	39.8
10A05B70L05K	50	170	70	5	0.1	**0.0**	46.2
10A05B70L10K	50	170	70	10	0.9	**0.0**	43.9
11A05B70L05K	55	188	70	5	0.5	**0.0**	46.2
11A05B70L10K	55	188	70	10	0.2	**0.1**	42.5
12A05B70L05K	60	206	70	5	0.5	**0.1**	43.3
12A05B70L10K	60	206	70	10	0.8	**0.1**	42.6

Quality of Lower Bounds

Instances from Konak (2012), EJOR

	Properties				LP gap [%]					
					type I			type II		
Instance	$\|V\|$	$\|E\|$	λ_{max}	$\|\mathcal{K}\|$	L_{MCF}	L_{CUT}	NA	L_{MCF}	L_{CUT}	NA
040N_05K_30L	40	396	30	5	**5.2**	**5.2**	39.5	**0.0**	**0.0**	75.1
040N_05K_35L	40	544	35	5	**5.6**	**5.6**	25.9	**0.4**	**0.4**	70.2
040N_10K_30L	40	396	30	10	7.8	**6.3**	41.0	**0.0**	**0.0**	74.0
040N_10K_35L	40	544	35	10	5.6	**4.6**	26.7	6.4	**4.3**	68.5
050N_05K_30L	50	558	30	5	1.3	**0.9**	32.2	**0.0**	**0.0**	71.7
050N_05K_35L	50	744	35	5	**0.0**	**0.0**	28.8	**0.0**	**0.0**	80.1
050N_10K_30L	50	558	30	10	12.2	**8.6**	48.1	**0.0**	**0.0**	76.0
050N_10K_35L	50	744	35	10	12.0	**9.0**	35.8	**0.0**	**0.0**	80.0
060N_05K_30L	60	610	30	5	**7.5**	**7.5**	51.1	**4.9**	**4.9**	82.8
060N_05K_35L	60	824	35	5	**0.0**	**0.0**	36.6	**0.0**	**0.0**	75.0
060N_10K_30L	60	610	30	10	**12.9**	**12.9**	50.1	**7.2**	**7.2**	79.7
060N_10K_35L	60	824	35	10	**3.6**	**3.6**	36.7	**0.0**	**0.0**	74.7
080N_05K_30L	80	1282	30	5	**0.0**	**0.0**	17.0	**1.4**	**1.4**	67.2
080N_05K_35L	80	1706	35	5	0.3	**0.0**	13.9	**0.0**	**0.0**	70.7
080N_10K_30L	80	1282	30	10	**1.2**	**1.2**	24.2	**0.5**	**0.5**	61.8
080N_10K_35L	80	1706	35	10	**3.2**	-	20.6	**0.0**	**0.0**	69.6
160N_05K_30L	160	5546	30	5	**0.0**	-	20.3	**1.4**	**1.4**	77.2
160N_05K_35L	160	7248	35	5	**0.5**	-	17.8	**3.2**	**3.2**	71.2
160N_10K_30L	160	5546	30	10	-	-	**31.1**	**2.5**	**2.5**	71.5
160N_10K_35L	160	7248	35	10	-	-	**28.4**	**0.5**	**0.5**	64.7

Speedup ratio to the NA model

CPU time(NA) / CPU time(algorithm)

	Speedup ratio			
Instance	B&P2 (Li et al.)	L_{MCF}	L_{CUT}-s	L_{CUT}-d
04A05B70L05K	**8.4**	6.2	3.1	3.2
04A05B70L10K	**55.9**	45.2	35.7	19.8
05A05B70L05K	20.8	**21.0**	15.7	14.1
05A05B70L10K	**107.3**	66.9	53.6	44.9
06A05B70L05K	7.1	**40.2**	25.4	22.3
06A05B70L10K	61.1	134.6	**164.4**	98.7
07A05B70L05K	31.8	**45.6**	19.2	16.2
07A05B70L10K	34.6	324.2	**476.3**	289.8
08A05B70L05K	9.3	**454.4**	216.3	81.7
08A05B70L10K	92.0	466.2	**543.7**	218.6
09A05B70L05K	9.9	**225.4**	110.9	69.0
09A05B70L10K	40.5	298.6	**391.6**	237.8
10A05B70L05K	40.9	**631.0**	319.8	122.1
10A05B70L10K	33.6	876.1	**1337.3**	683.9
11A05B70L05K	25.1	**540.6**	306.5	123.0
11A05B70L10K	45.4	954.9	**1500.4**	555.7
12A05B70L05K	5.2	**716.9**	528.4	215.0
12A05B70L10K	110.1	755.1	**1164.3**	405.9

L-MCF vs. L-CUT

Directed Konak instances, type II (inversely correlated distance)

Instance	gap [%]				time [s]			
	L_{MCF}	L_{CUT}-s	L_{CUT}-d	NA	L_{MCF}	L_{CUT}-s	L_{CUT}-d	NA
040N_05K_30L	0.0	0.0	0.0	0.0	< 1	< 1	1	8
040N_05K_35L	0.0	0.0	0.0	0.0	< 1	2	3	25
040N_10K_30L	0.0	0.0	0.0	0.0	< 1	2	3	844
040N_10K_35L	0.0	0.0	0.0	0.0	2	26	19	2862
050N_05K_30L	0.0	0.0	0.0	0.0	< 1	1	1	44
050N_05K_35L	0.0	0.0	0.0	0.0	< 1	2	4	2761
050N_10K_30L	0.0	0.0	0.0	57.3	1	4	27	7200
050N_10K_35L	0.0	0.0	0.0	66.4	2	26	16	7200
060N_05K_30L	0.0	0.0	0.0	41.6	3	22	49	7200
060N_05K_35L	0.0	0.0	0.0	0.0	< 1	1	4	1282
060N_10K_30L	0.0	0.0	0.0	70.5	37	1534	1093	7200
060N_10K_35L	0.0	0.0	0.0	60.8	2	24	9	7200
080N_05K_30L	0.0	0.0	0.0	44.6	3	20	24	7200
080N_05K_35L	0.0	0.0	0.0	41.6	8	37	15	7200
080N_10K_30L	0.0	0.0	0.0	41.0	22	88	45	7200
080N_10K_35L	0.0	0.0	0.0	54.9	39	320	39	7200
160N_05K_30L	0.0	0.0	0.0	72.3	258	2515	412	7200
160N_05K_35L	0.0	0.0	0.0	64.9	854	6266	616	7200
160N_10K_30L	0.0	46.1	0.0	68.4	2022	7200	2612	7200
160N_10K_35L	0.0	50.8	0.0	62.5	4283	7200	2502	7200

Conclusion

- significantly beats state-of-the-art
- very strong LP bounds
- our algorithms find optimal solutions for instances with:
  - 160 vertices and more than 7000 arcs

Future Work

- Network Design with Relays Under Uncertainty (robust or stochastic models?)
- Applications:
  - Telecommunications: Quantum-Key-Distribution (QKD), placing of encryption keys along the network, so as to make sure each O-D path is encrypted according to the Quantum Computing technology.
  - E-mobility: maximum number of recharging stops, distance limits for the trips?

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